



Certified bottleneck transitions and balanced capacity upgrades in robust production–distribution networks

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Abstract

Manufacturing resources and shipping routes can constrain the same customer commitment, making both uncertainty audits and capacity spending sensitive to changing bottlenecks. This paper develops a certified production–distribution analysis under a static uncertainty model with common resource-coefficient inflation. A reparameterization converts the uncertainty path to a fixed-matrix LP and yields a finite dual-line envelope for maximal supportable commitment stress. A certificate-driven interval procedure traces transitions with explicit upper and lower bounds. A budgeted resource–route upgrade model then derives marginal budget value by minimizing the largest price-normalized multiplier over the entire baseline-optimal dual face. A family with k simultaneous bottlenecks shows that an arbitrary extreme certificate can overstate marginal value by a factor k , while upgrading only its indicated bottleneck produces zero gain. An explicit two-plant family provides an exact switching law and constructive shipment proof. Computation covers 36 networks with up to 144 shipment variables, 180 budgeted upgrade cases and 25 serial-bottleneck cases. Independently solved primal and dual programs verify the envelope and spending calculations. The method specializes established LP and parametric-optimization principles; its contribution is the linked bottleneck-and-upgrade audit with explicit failure families. All experiments are synthetic.

Keywords: integrated production–distribution; robust feasibility; demand stress; bottleneck switching; dual certificate; industrial mathematics.

1. Introduction and positioning

Production and distribution are often optimized sequentially: select a profitable manufacturing mix, then allocate shipments. This sequence can fail when a route capacity blocks a customer commitment or when delivering a chosen mix consumes a different pattern of inputs across plants. A useful audit should locate the largest supportable commitment increase and identify whether resources or routes are the limiting factor.

The four selected application works supply the starting points. Paint production [21] and medication manufacture [23] demonstrate resource-constrained production formulations. Harmonic-mean transportation [20] supplies a distribution heuristic, whose feasibility and cost must be distinguished from integrated optimality. The sensitivity application [22] motivates declaring the parameter and response being varied. In this paper, those quantities are an industrial uncertainty radius, a customer-commitment stress and an optimized shipment cost. No epidemic dynamics are used.

Robust multi-site production is already represented in Leung et al. [15], and multi-product, multi-site aggregate planning under uncertainty is addressed by Mirzapour Al-e-hashem et al. [16]. Consequently, neither the phrase integrated planning nor the use of uncertainty constitutes a novelty claim. The present study develops a general fixed-matrix bottleneck envelope under common coefficient inflation, links it to balanced resource–route budget upgrades, and proves a dimension-dependent failure of using one extreme baseline certificate. The small exact family supplies a constructive proof example; larger networks test the general audit.

2. General static robust formulation

Let i index plants, p products, j customer warehouses and k resource types. Let $y_{ipj} \geq 0$ be a shipment and $x_{ip} = \sum_j y_{ipj}$ the required production. The resource-use coefficient a_{ikp} and available resource b_{ik} satisfy

$$a_{ikp} \in [a_{ikp}^0, a_{ikp}^0 + \theta \hat{a}_{ikp}], \quad b_{ik} \in [b_{ik}^0 - \theta \hat{b}_{ik}, b_{ik}^0], \quad (1)$$

with nonnegative perturbation directions and fixed $\theta \geq 0$. Route capacities are $U_{ipj} \geq 0$. Commitments range between d_{pj}^0 and $d_{pj}^0 + \rho \Delta_{pj}$, with $d^0, \Delta \geq 0$ and stress $\rho \geq 0$. The uncertainty set is a rectangular box; in particular, simultaneous worst endpoints are admissible.

The plan is selected before realization. It must produce enough for every commitment realization, and surplus at smaller demand realizations may remain in storage. Thus the model is a static reserve plan, not a model that forbids any surplus in every realization. If exact realization-by-realization delivery without storage is required, adjustable decisions and a separate recourse model are needed.

Write $a^+ = a^0 + \theta \hat{a}$ and $b^- = b^0 - \theta \hat{b}$. The static stress problem is

$$\max_{y, \rho} \quad \rho, \quad (2)$$

$$\sum_p a_{ikp}^+ \sum_j y_{ipj} \leq b_{ik}^-, \quad \text{all } i, k, \quad (3)$$

$$\sum_i y_{ipj} \geq d_{pj}^0 + \rho \Delta_{pj}, \quad \text{all } p, j, \quad (4)$$

$$0 \leq y_{ipj} \leq U_{ipj}, \quad \rho \geq 0. \quad (5)$$

For a nonnegative static plan, these endpoints are necessary and sufficient for all realizations in the box. Necessity follows by choosing the simultaneous adverse endpoint scenario. Sufficiency follows from monotone resource consumption and monotone commitments. The endpoint reduction is standard robust optimization [25, 3]; alternative uncertainty budgets [4] need their own counterpart.

For maximal feasible stress or cost minimization with positive unit costs, shipments can be reduced to satisfy the upper commitments exactly. Reducing any surplus shipment improves or preserves resource and route feasibility. Hence the inequality formulation has no artificial advantage from surplus. We assume the base commitments are feasible and the stress optimum is finite. Without base feasibility, an empty feasible set must be reported rather than setting the optimum to zero.

3. A general upper certificate for commitment stress

Let R map the shipment vector to resource consumption, let D map it to product–customer totals, and flatten U, b^-, d^0, Δ consistently. Then (2) is

$$Ry \leq b^-, \quad Dy \geq d^0 + \rho \Delta, \quad y \leq U, \quad y, \rho \geq 0.$$

Theorem 3.1 (Weighted bottleneck certificate). *Let $\lambda, \mu, \nu \geq 0$ satisfy*

$$R^T \lambda + \nu - D^T \mu \geq 0, \quad \Delta^T \mu \geq 1. \quad (6)$$

Every feasible stress satisfies

$$\rho \leq (b^-)^T \lambda + U^T \nu - (d^0)^T \mu. \quad (7)$$

If a feasible plan attains this bound, it certifies the maximal stress. Under the stated feasible finite LP assumptions, an optimal certificate exists.

Proof. The demand constraints imply $\rho \Delta^T \mu \leq \mu^T D y - \mu^T d^0$. The coefficient inequality and $y \geq 0$ give $\mu^T D y \leq \lambda^T R y + \nu^T y$. Resource and route bounds then give $\rho \Delta^T \mu \leq \lambda^T b^- + \nu^T U - \mu^T d^0$. Since $\rho \geq 0$ and $\Delta^T \mu \geq 1$, (7) follows. Equality against a feasible plan establishes optimality by weak duality. The minimization of the right-hand side under (6) is the LP dual, so strong duality gives the last statement [9, 5, 26]. $\square$

The certificate weights reveal how a resource or a group of route caps limits commitments. They need not be unique. A zero weight does not prove that a constraint is operationally unimportant; degeneracy can allow more than one optimal certificate. Convex [6] and polyhedral [24] foundations help distinguish this issue from numerical tolerance, which must be checked separately [19].

4. An exact two-plant switching family

There are two plants, two products and three customers. Every plant has nominal resource-use matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

At radius $0 \leq \theta \leq 1$, the worst resource matrix is sA with $s = 1 + 0.1\theta$. Worst input capacities are

$$B^-(\theta) = \begin{pmatrix} 120 - 20\theta & 100 - 20\theta \\ 100 - 20\theta & 120 - 20\theta \end{pmatrix}, \quad (8)$$

where rows are plants and columns are resource types. Define effective capacities

$$H = \frac{120 - 20\theta}{s}, \quad L = \frac{100 - 20\theta}{s}.$$

Commitments and their stress directions are

$$d^0 = \begin{pmatrix} 15 & 20 & 10 \\ 20 & 10 & 15 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 5 & 8 & 4 \\ 6 & 5 & 7 \end{pmatrix}. \quad (9)$$

Rows are products and columns are customers. Product A route caps at plants 1 and 2 are (20, 14, 20) and (20, 16, 20). Every product B route cap is 20. These inputs are synthetic normalized quantities.

Define

$$R(\theta) = \frac{(220 - 40\theta)/s - 135}{53}. \quad (10)$$

Theorem 4.1 (Exact resource–route switching law). *For the specified family,*

$$\rho^*(\theta) = \min\{1.25, R(\theta)\}, \quad 0 \leq \theta \leq 1. \quad (11)$$

The limiting certificate changes from the product A route to customer 2 to the aggregate second resource at

$$\theta_c = \frac{18.75}{60.125} = \frac{150}{481} \approx 0.3118503. \quad (12)$$

4.1 Upper bounds

The two routes for product A to customer 2 have total capacity 30, whereas the required quantity is $20 + 8\rho$. Thus $\rho \leq (30 - 20)/8 = 1.25$. It is an explicit certificate: set the corresponding demand weight and two route weights to $1/8$, all other weights to zero.

Total product quantities are $a = 45 + 17\rho$ and $b = 45 + 18\rho$. They use $a + 2b = 135 + 53\rho$ units of effective second resource. The two plants jointly have $H + L = (220 - 40\theta)/s$. Hence $\rho \leq R(\theta)$. A normalized certificate assigns demand weights $1/53$ to every product A demand and $2/53$ to every product B demand, assigns second-resource weights $1/(53s)$ at both plants, and assigns all other weights zero. Its Δ normalization is exactly one.

An aggregate necessary inequality alone would not establish feasibility. The next lemma and construction prove that the smaller bound is attainable.

Lemma 4.2 (Two-source shipment disaggregation). *For one product with demands $d_j \geq 0$, source 1 quantity x_1 , total production $\sum_j d_j$, and route caps U_{1j}, U_{2j} , a shipment with exact customer totals exists if and only if*

$$d_j \leq U_{1j} + U_{2j} \quad \forall j, \quad \sum_j \max(0, d_j - U_{2j}) \leq x_1 \leq \sum_j \min(U_{1j}, d_j). \quad (13)$$

Proof. The source 1 shipment to customer j must lie in the interval $[\max(0, d_j - U_{2j}), \min(U_{1j}, d_j)]$. Those intervals are nonempty precisely when the individual combined-capacity inequalities hold. Their possible sums fill the interval between the sum of lower and sum of upper endpoints. A greedy filling procedure starting from the lower endpoints achieves any sum in that interval; source 2 supplies the complementary quantities. $\square$

4.2 Constructive lower bound and switching proof

Completion of Theorem 4.1. Choose any $0 \leq \rho \leq \min\{1.25, R(\theta)\}$. Set $\delta = 10/s$ and allocate production as

$$x_{1A} = a/2 + \delta, \quad x_{1B} = b/2 - \delta, \quad x_{2A} = a/2 - \delta, \quad x_{2B} = b/2 + \delta. \quad (14)$$

All quantities are positive. Since $b \geq a$ and $a + 2b \leq H + L$, both plant resource inequalities hold. For example, plant 1's second-resource use is $a/2 + b - \delta \leq (H + L)/2 - (H - L)/2 = L$. Plant 2's second-resource use is at most H . The first-resource uses are no larger than the corresponding second-resource expressions with a, b exchanged, so they are at most H and L , respectively.

For product A, the combined route inequalities hold because its middle demand is at most 30 and the others are below 40. The sum of lower endpoints in (13) is $4 + 8\rho + \max(0, 5\rho - 5)$, at most 15.25, whereas $x_{1A} \geq 22.5 + 10/1.1 > 31$. The sum of upper endpoints minus x_{1A} is $16.5 + 0.5\rho - \delta$ for $\rho \leq 1$, and $21.5 - 4.5\rho - \delta$ for $1 \leq \rho \leq 1.25$. Both are positive, the latter being at least 5.875.

For product B, each combined route cap is 40 and every demand is at most 27.5. The lower-endpoint sum is $6\rho + \max(0, 7\rho - 5)$, at most 11.25. But $x_{1B} \geq 12.5$. The upper-endpoint sum is at least 45, whereas $x_{1B} \leq 33.75$. Therefore both products admit exact shipments by Lemma 4.2. The construction proves the entire lower bound, including its endpoint.

Finally, solving $R(\theta) = 1.25$ gives $220 - 40\theta = 201.25(1 + 0.1\theta)$ and hence (12). The function R is decreasing, so the route is limiting before that radius and resources are limiting after it. $\square$

The construction is special to this family. It does not assert that aggregate resource inequalities and singleton route checks are sufficient for every multi-plant, multi-product network. General systems need the LP certificate or a more complete network cut analysis.

5. Delivered-cost certificates at feasible stress

Having found a feasible stress level, consider minimizing delivered cost. Let $k_{ipj} = g_{ip} + c_{ipj}$ include unit manufacturing and shipping costs. With exact worst commitments $d = d^0 + \rho\Delta$, solve

$$C^*(\rho, \theta) = \min\{k^T y : Ry \leq b^-, Dy = d, 0 \leq y \leq U\}. \quad (15)$$

This is a distinct objective from maximizing stress. A plan that attains ρ^* need not minimize its cost. A cost certificate consists of resource weights $u \leq 0$, unrestricted demand potentials v , and route weights $w \leq 0$ satisfying

$$R^T u + D^T v + w \leq k. \quad (16)$$

Every feasible shipment then satisfies

$$k^T y \geq (b^-)^T u + d^T v + U^T w. \quad (17)$$

Indeed, multiply (16) by nonnegative y , use $Dy = d$, and note that multiplying resource and route upper bounds by nonpositive weights reverses their inequalities. Equality provides optimality. This extends the familiar transportation-potential certificate [12, 1, 10] to resource and route caps; it is a standard LP dual specialization.

Stepping-stone improvement [8] concerns the simpler transportation structure. The harmonic-mean rule [20] is not an exact method for the integrated model because source supplies themselves depend on production. Similarly, one-to-one lot assignments can use Kuhn [14], Munkres [17] or the methods surveyed by Burkard et al. [7]; these assignment subproblems must not replace the coupled resource constraints.

6. Synthetic computational study

The script computes maximal stress for 101 radii between zero and one, compares each LP optimum with (11), constructs the production split (14), and greedily disaggregates both products. It checks every resource row, route cap and customer total. It also solves the general stress dual independently and compares its objective with the primal optimum.

For the delivered-cost experiment, manufacturing costs are (3, 4) at plant 1 and (4, 3) at plant 2. Shipping matrices are

$$C_1 = \begin{pmatrix} 1 & 3 & 5 \\ 4 & 2 & 3 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 4 & 2 \end{pmatrix}.$$

These costs are synthetic positive accounting units. The model is not calibrated to a factory or customer network. Cost potentials returned by the solver are verified against (16) and the primal–dual objective gap.

Table 1: *Selected synthetic cases, independently checked by linear programming.*

θ	ρ^*	Minimum delivered cost
0	1.25	741
0.25	1.25	746.884
0.31185	1.25	750.395
0.5	1.04672	697.195
0.75	0.787626	638.418
1	0.540309	587.063

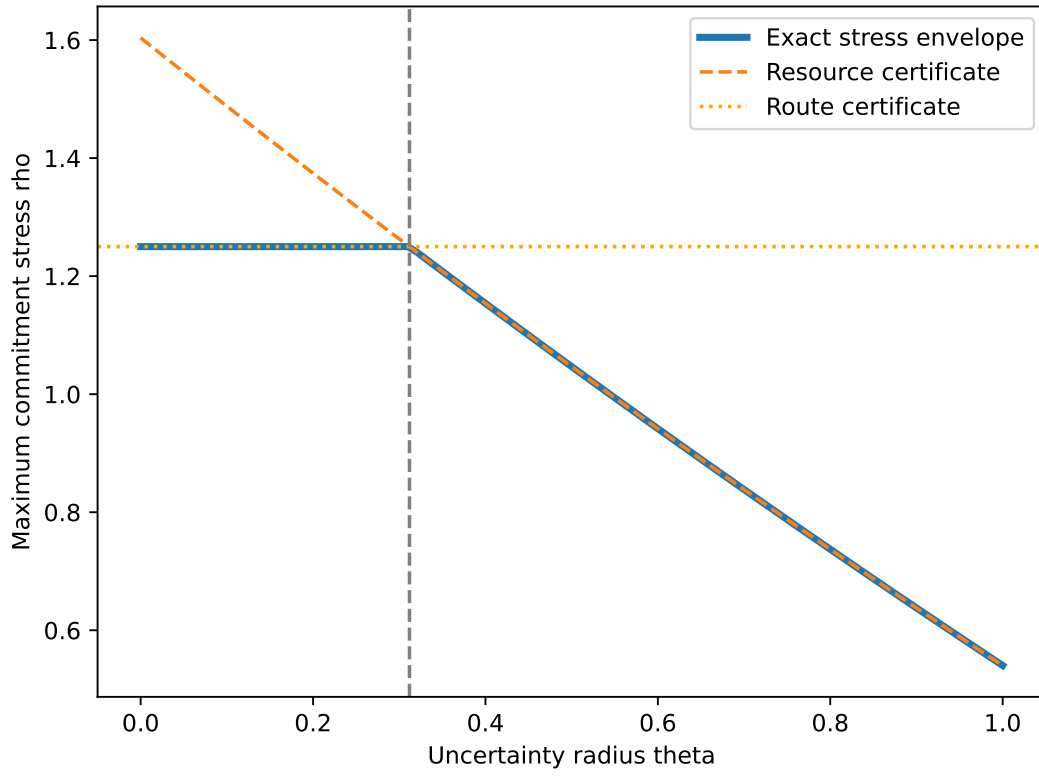


Figure 1: *Exact maximal commitment stress and its separate route and resource upper certificates. Their intersection is a proved switching radius.*

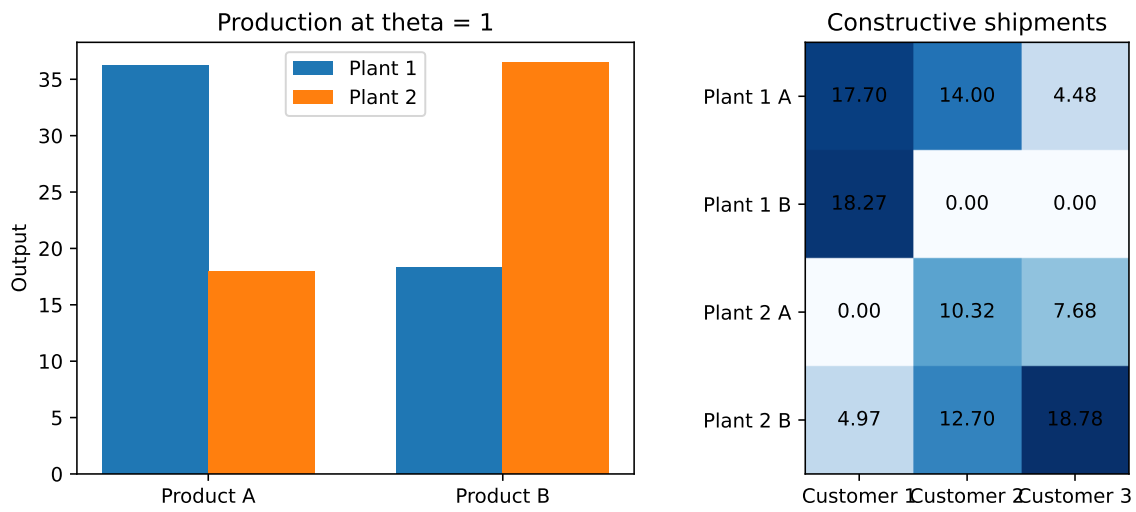


Figure 2: *A constructive production split and customer shipments at the selected stressed endpoint. All inputs are synthetic industrial quantities.*

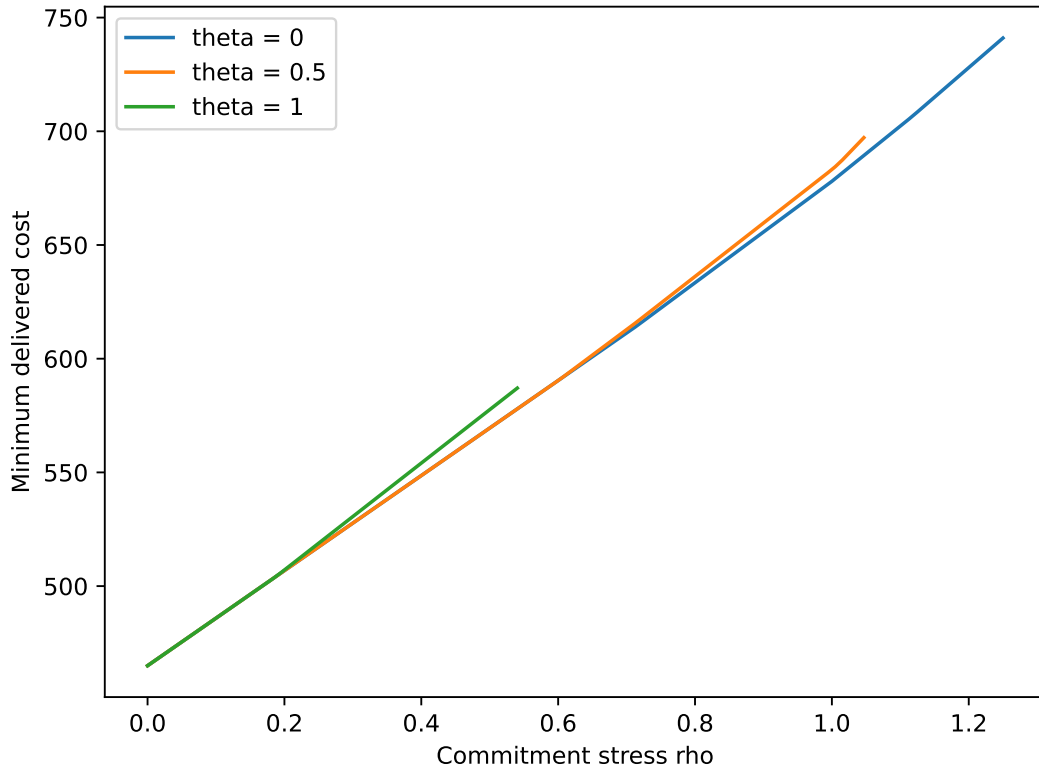


Figure 3: Minimum delivered cost across feasible stress levels at selected uncertainty radii. Curves stop at their own feasibility thresholds.

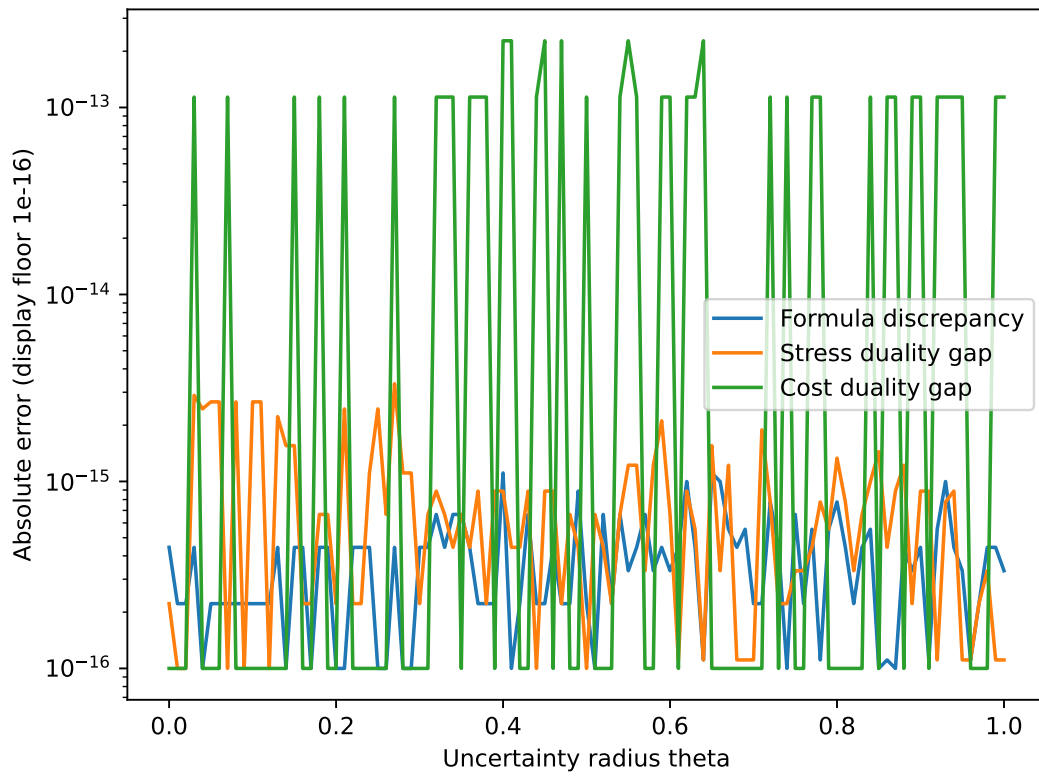


Figure 4: Primal-dual gaps and analytic-formula discrepancies across the radius grid. Plotted values have a display floor; exact numerical checks use the unfloored values.

7. General uncertainty paths and bottleneck envelopes

The explicit two-plant family is a proof example, not the limit of the model. Consider arbitrary finite numbers of plants, products, customers and resource rows. Let $R_0 \geq 0$ be the shipment-to-resource matrix, D the customer-total incidence matrix, $U \geq 0$ route caps, $d^0 \geq 0$ baseline commitments and $\Delta \geq 0$ their increase direction. Assume at least one positive component of Δ , feasibility at the baseline, and a finite attained stress optimum on the full analysis interval.

Suppose the worst resource matrix and capacities have the structured form

$$R(\theta) = (1 + h\theta)R_0, \quad b^-(\theta) = b^0 - \theta d_b, \quad h \geq 0, \quad d_b \geq 0. \quad (18)$$

The common coefficient factor is an explicit restriction. General independent coefficient perturbations do not admit the following single reparameterization. Define

$$z = \frac{\theta}{1 + h\theta}, \quad e_b = d_b + hb^0. \quad (19)$$

Then $\theta = z/(1 - hz)$ on the valid domain and

$$\frac{b^0 - \theta d_b}{1 + h\theta} = b^0 - ze_b. \quad (20)$$

The uncertainty path therefore becomes a fixed-matrix right-hand-side parametric LP:

$$F(z) = \max\{\rho : R_0 y \leq b^0 - ze_b, \quad Dy \geq d^0 + \rho\Delta, \quad 0 \leq y \leq U, \rho \geq 0\}. \quad (21)$$

Theorem 7.1 (Finite bottleneck envelope under common coefficient inflation). *Let $\mathcal{Q}$ be the vertices of the fixed dual-feasible polyhedron*

$$\lambda, \mu, \nu \geq 0, \quad R_0^T \lambda - D^T \mu + \nu \geq 0, \quad \Delta^T \mu \geq 1.$$

On every feasible bounded parameter interval,

$$F(z) = \min_{q \in \mathcal{Q}} \{a_q - z\beta_q\}, \quad a_q = (b^0)^T \lambda + U^T \nu - (d^0)^T \mu, \quad \beta_q = e_b^T \lambda \geq 0. \quad (22)$$

Thus F is nonincreasing, continuous, concave and piecewise affine in z . In θ , each active branch is fractional affine, of the form $(a_q + (ha_q - \beta_q)\theta)/(1 + h\theta)$. A transition occurs when two active certificate lines meet; merely intersecting two arbitrary feasible certificates need not locate an actual transition.

Proof. The effective-capacity identity follows by substituting z . The dual feasible set is independent of z and lies in the nonnegative orthant, so it is pointed. A finite attained dual optimum has an optimal vertex. Strong duality gives the finite lower envelope in (22). Nonnegative e_b, λ give nonpositive slopes. Standard finite-envelope properties prove continuity, concavity and piecewise affinity. Substitution back into θ gives the displayed fractional form. $\square$

A certificate with positive resource weights and positive route weights identifies a mixed obstruction. Calling every active certificate either purely a route bottleneck or purely a resource bottleneck would conceal this case. The algorithm reports both weight groups. Under degeneracy there may be several optimal certificates, so a reported support is one valid explanation rather than a unique physical diagnosis. Parametric LP and genuine shadow-price analysis already address these issues in broader settings [11, 13, 18].

8. An interval-wide algorithm with numerical certificates

At a parameter z , solve the stress primal and its dual independently. Check both feasibility systems and their objective gap. The dual returns a globally valid line $a_q - z\beta_q$ touching F at that point. On an interval $[a, b]$, the minimum of endpoint certificate lines is an upper bound and the chord of endpoint optimal values is a lower bound, because F is concave. Their greatest difference is found by checking the endpoints and the certificate-line intersection.

Query at that intersection whenever the interval bound exceeds the specified tolerance η , then recurse. If the two certificates coincide as affine lines, the chord and upper bound coincide and the full interval is already solved. If a query reaches the certificate intersection exactly, concavity and the upper supporting lines prove both resulting subintervals affine. Otherwise the query supplies an additional active certificate. This is a specialization of the classical intersection-query approach to parametric optimization [2]; no new general dichotomic principle is claimed.

The final record contains parameter knots, dual lines and interval gap bounds. For each terminal interval,

$$0 \leq F(z) - L_{ab}(z) \leq \eta \quad (a \leq z \leq b). \quad (23)$$

In exact arithmetic, extreme-point certificates give finite output-dependent termination because there are finitely many active affine branches. This is not a polynomial bound in network size: the number of regions can be large [18]. In floating point, the implementation checks oracle residuals and reports them separately; it does not claim a formally verified rational certificate. A dense grid supplies independent numerical comparisons but has no interval-wide guarantee by itself.

9. Budgeted upgrades and balanced shadow-price interpretation

At a fixed uncertainty radius, use effective resource capacities $\bar{b}$, fixed matrix R_0 and route caps U . Let $r \geq 0$ purchase effective resource capacity and $w \geq 0$ purchase route capacity. Their positive unit prices are γ and ξ , and the available upgrade budget is $B \geq 0$. The upgrade problem is

$$\Phi(B) = \max \left\{ \rho : \begin{array}{ll} R_0 y - r \leq \bar{b}, & Dy \geq d^0 + \rho \Delta, \\ y - w \leq U, & \gamma^T r + \xi^T w \leq B, \\ y, r, w, \rho \geq 0 \end{array} \right\}. \quad (24)$$

These prices refer to effective resource units. A physical capacity price must be converted consistently by the factor $1 + h\theta$. No fixed-charge construction costs or integer projects are represented.

Its dual has the previous stress-certificate constraints and an additional multiplier $\kappa \geq 0$ satisfying

$$\lambda_i \leq \kappa \gamma_i \quad \forall i, \quad \nu_e \leq \kappa \xi_e \quad \forall e, \quad (25)$$

with objective

$$\bar{b}^T \lambda + U^T \nu - (d^0)^T \mu + B\kappa. \quad (26)$$

The inequalities follow from the coefficients of r and w in the LP dual. Equality of primal and dual objectives certifies both the supported stress and the spending decision.

Theorem 9.1 (Marginal budget value over the whole optimal dual face). *Assume (24) is feasible and bounded for budgets in a right neighbourhood of zero. Let $\mathcal{Q}_0$ be all feasible stress dual triples attaining $\Phi(0)$ at $\bar{b}, U$. Then*

$$\Phi'_+(0) = \min_{(\lambda, \mu, \nu) \in \mathcal{Q}_0} \max \left\{ \max_i \frac{\lambda_i}{\gamma_i}, \max_e \frac{\nu_e}{\xi_e} \right\}. \quad (27)$$

The minimum is obtained by an LP minimizing κ subject to dual feasibility, $\lambda \leq \kappa \gamma$, $\nu \leq \kappa \xi$, and the baseline objective equality. An arbitrary optimal dual triple yields a valid upper bound on the right

derivative, but need not give its exact value.

Proof. For a fixed stress triple, the smallest admissible budget multiplier is the maximum of the stated ratios. The augmented dual is a pointed polyhedron with finitely many vertices; on a feasible bounded budget interval, $\Phi(B)$ is the minimum of their affine objectives. Its right derivative at zero is the smallest slope among baseline-active vertices. Minimizing the multiplier over the baseline-active face is equivalent to selecting that smallest slope. Conversely every triple in $\mathcal{Q}_0$ extends to a feasible augmented dual by choosing its maximum ratio. This proves (27) and the auxiliary-LP formulation. Keeping any one such feasible dual gives $\Phi(B) \leq \Phi(0) + B\kappa$, establishing the last claim. $\square$

9.1 Why one shadow-price vector can misdirect spending

Consider two serial bottlenecks, each with capacity one. Continuous purchases r_1, r_2 cost one per unit and satisfy $r_1 + r_2 \leq B$. The largest supported throughput obeys $F \leq 1 + r_1$ and $F \leq 1 + r_2$, hence

$$\max F = 1 + B/2, \quad (28)$$

attained by $r_1 = r_2 = B/2$. At zero budget, every nonnegative dual pair with $\lambda_1 + \lambda_2 = 1$ is optimal. An extreme certificate $(1, 0)$ has maximum price ratio one, whereas the minimum over the face is one-half at $(1/2, 1/2)$. Spending the full budget on the first bottleneck gives no throughput gain at all. The example demonstrates both an overstated marginal budget value and a failed greedy allocation. It illustrates known degeneracy concerns [13]; its role is to validate the balanced face calculation in the integrated model, not to claim discovery of degenerate LPs.

Corollary 9.2 (Dimension-dependent overstatement and a zero-gain greedy rule). *For one shipment with $k - 1$ unit-capacity resource rows and one unit-capacity route, let every unit upgrade cost one. Baseline demand is zero and the stress direction is one. Then $\Phi(B) = 1 + B/k$, the balanced marginal budget value is $1/k$, and an extreme baseline stress certificate can give the upper slope one. Its overstatement factor is k . For $k \geq 2$, spending the entire budget on that certificate's single bottleneck produces zero gain while balanced spending produces B/k .*

Proof. All k capacities must accommodate the same shipment. Summing $\rho \leq 1 + r_i$ gives $k\rho \leq k + B$, attained by $r_i = B/k$. Baseline-optimal certificate weights form the probability simplex. Its smallest maximum coordinate is $1/k$, while each extreme point has maximum coordinate one. Leaving any other capacity at one limits single-bottleneck spending to the baseline shipment. $\square$

For finite budgets the marginal value can change as bottlenecks switch. The full upgrade LP must therefore be solved, or its budget envelope traced with the same interval algorithm. Formula (27) concerns a right neighbourhood of zero and is not a licence to extrapolate a local slope indefinitely.

10. Larger networks and upgrade experiments

The supplemental study generates 36 networks, 12 each with (plants, products, customers, resources per plant) equal to $(3, 2, 4, 2)$, $(5, 3, 6, 2)$ and $(8, 3, 6, 3)$. Their shipment-variable counts are 24, 90 and 144. Positive resource coefficients, commitment directions, route caps and capacities are generated around a feasible baseline shipment so that every radius from zero to one remains baseline-feasible. Coefficients inflate with $h = 0.1$, and resource capacities lose positive fractions of their nominal values. Full inputs are archived.

Each network is traced with the dual-line interval method and independently solved at 101 uniform uncertainty radii. Comparison checks stress values, every primal constraint, every dual inequality and the terminal interval gap. The explicit family in Theorem 4.1 remains an exact test of the switching radius. Separate experiments solve both upgrade primal and augmented dual for budgets 0, 1, 2, 5

and 10 at radius 0.5. The auxiliary face LP computes the zero-budget marginal value; a small positive budget provides a numerical right-derivative check. The serial-bottleneck example verifies that the balanced value is 0.5 while an arbitrary extreme certificate can suggest one.

This study evaluates general-network auditing and resource-route spending under the specified continuous model. It does not establish factory savings, accuracy of uncertainty estimates, a polynomial tracing complexity or a speed advantage over specialized parametric solvers. The principal evidence is independently matched primal/dual objectives and an interval-wide envelope bound, supplemented by transparent synthetic comparisons.

Table 2: *Expanded synthetic study: measured numerical summaries.*

Metric	Result
Network instances	36
Maximum shipment variables	144
Mean tracing oracle queries	7.63889
Budgeted upgrade cases	180
Maximum stress primal–dual residual	2.05738e-10
Maximum upgrade primal–dual gap	1.61871e-13

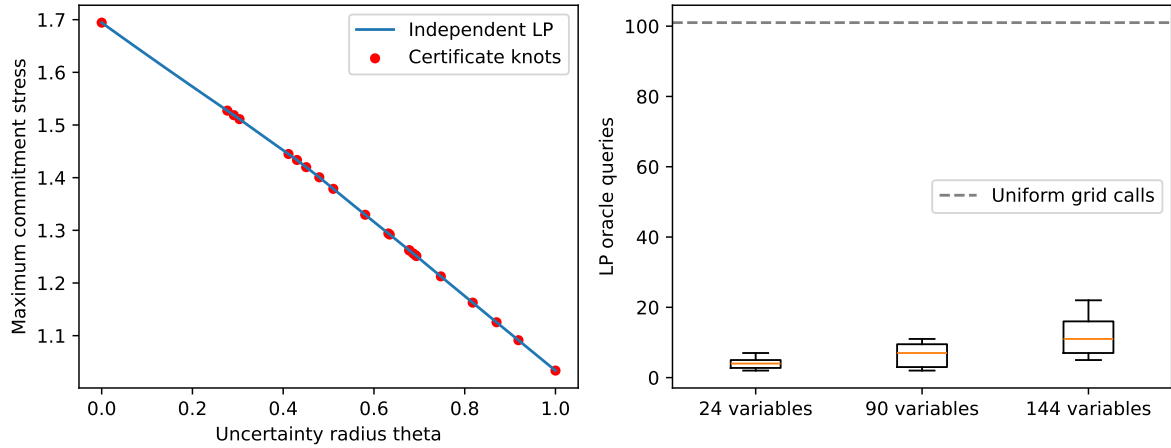


Figure 5: *A general-network uncertainty envelope and oracle-query counts across the three network-size groups.*

11. Positioning and industrial limitations

Robust production planning is established [15, 16], as are endpoint uncertainty sets [3, 25, 4], parametric LP [11, 18], and shadow-price ambiguity under degeneracy [13]. The line-intersection tracing principle also has prior parametric-optimization treatments [2]. These foundations are explicitly attributed.

The proposed contribution links the common-inflation bottleneck envelope to a resource-route upgrade dual, a balanced optimal-face marginal budget calculation and a dimension-dependent zero-gain greedy failure family. The exact two-plant switching construction is an independently proved example, rather than the sole claimed result. No claim of inventing robust planning, dual-face sensitivity or generic envelope tracing is made. Worldwide novelty of the specific combination remains subject to specialist review.

The model assumes divisible quantities, a static uncertainty box, fixed route choices and no setup costs or lead times. The uncertainty reparameterization requires common coefficient inflation; independently varying coefficients are outside that tracing theorem. Inventory or recourse is required if adverse-endpoint shipments cannot be retained at lower demand. Upgrade prices and capacity units must be

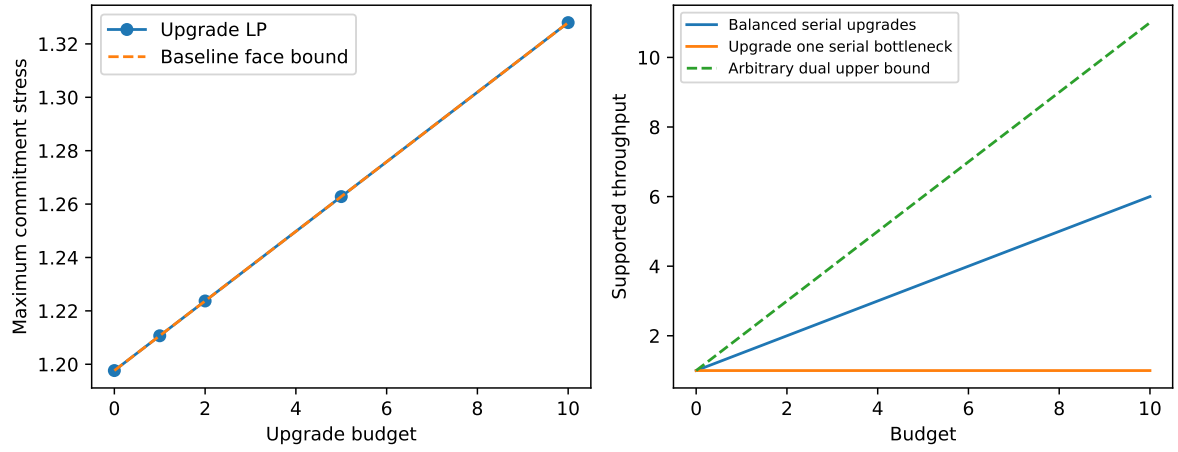


Figure 6: Budgeted upgrades compared with the baseline dual-face upper bound, and the two-bottleneck failure of one-sided spending.

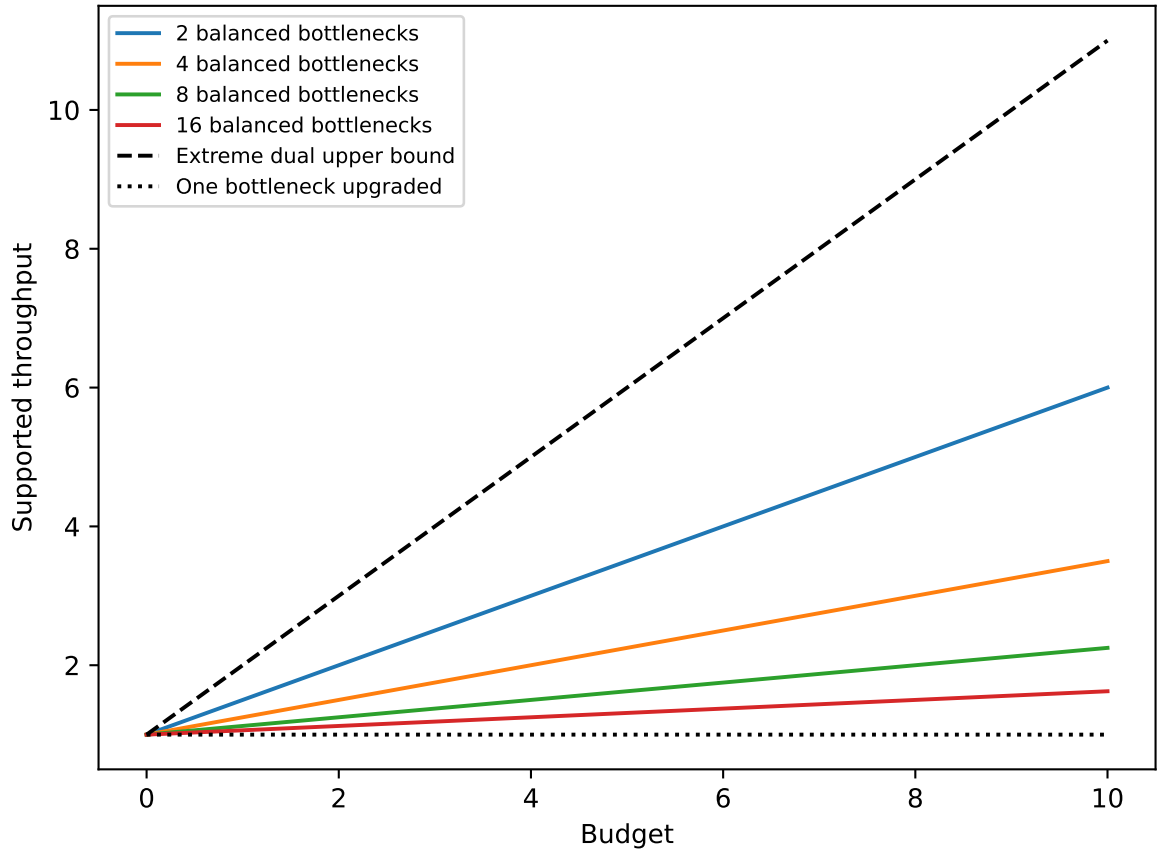


Figure 7: With k simultaneous bottlenecks, balanced spending gives a gain B/k while a single-bottleneck purchase gives no gain.

calibrated in a real network. Floating-point residual checks are not formal rational verification, and the number of parametric regions need not be polynomial.

12. Conclusion

The analysis connects uncertainty-induced bottleneck transitions with capacity-spending decisions. Common coefficient inflation admits a fixed-matrix certificate envelope, while budgeted upgrades require consideration of the full optimal dual face when the baseline is degenerate. An arbitrary extreme shadow-price vector can overstate marginal value by an unbounded dimension-dependent factor and direct spending that gives no gain. Exact constructions and independent larger-network primal/dual checks make these conclusions auditable within the static continuous model.

A. Reproduction and review

The archive supplies LaTeX sources, compiled PDF, figures in PDF and PNG, model inputs, CSV results, numerical summaries and the experiment script. Run `bash reproduce.sh`. All results are synthetic. Author review and a wider specialist literature comparison remain necessary before a priority claim or journal submission.

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